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## LOCAL-SOLUTION APPROACH TO QUASISTATIC RATE-INDEPENDENT MIXED-MODE DELAMINATION

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**Abstract.** *The quasistatic rate-independent evolution of a delamination at small strains in the so-called mixed mode, i.e. distinguishing opening (Mode I) from shearing (Mode II), devised in [40, 41], is rigorously analyzed in the context of a concept of stress-driven local solutions. The model has separately convex stored energy and is associative, namely the 1-homogeneous potential of dissipative force driving the delamination depends only on rates of internal parameters. An efficient fractional-step-type semi-implicit discretisation in time is shown to converge to (specific, stress-driven like) local solutions that may approximately obey the maximum-dissipation principle. Making still a spatial discretisation, this convergence as well as relevancy of such solution concept are demonstrated on a nontrivial 2-dimensional example.*

### 1. INTRODUCTION

Adhesive contacts represent an important area in contact mechanics and have numerous and continuously increasing applications. The process of damaging the adhesive surfaces between bulk materials is frequently referred to as *delamination* or debonding. It is observed experimentally that sometimes, or rather typically, more (or even substantially more) energy is needed in order for a delamination to occur in the so-called Mode II (shear) than the respective energy for a delamination in the so-called Mode I (opening). In general when delamination proceeds in a mixed (and a-priori not known) mode, we need a model which is sensitive to modes of delamination.

In this work, we use the Frémond's concept analogous to a bulk damage, assuming that the description of the damage is succeeded through a scalar variable, which is defined along the adhesive interfaces, taking values in the interval  $[0, 1]$  with 0 having the meaning of complete damage of the adhesive while 1 meaning complete operation of the adhesive, that is no damage appeared. Moreover, we will consider the adhesive to have some elastic response, also referred to as an imperfect or weak interface, opposite to the rigid/ideally-brittle adhesive interface. Moreover, we will confine ourselves to small strains and linearly responding materials in the bulk. Actually, the present model, in its simplest form, would correspond, following the classification in [22], to the so-called "initially elastic Barenblatt model" whose interface energy is given by a convex quadratic function of displacement

jump. It differs from the classical Griffith model, which is not adequate for predicting onset of delamination.

Let us remark that there is also another engineering model which phenomenologically prescribes energy needed for delamination dependent on the state (which is sometimes called “non-associative” model) through the ratio of tangential and normal stresses or displacements (whose  $\arctan$  is called a fracture-mode-mixity angle, see e.g. [2, 14, 17, 20, 47]). Mathematical justification of such a model seems possible only if a visco-elastic material with enough dissipative rheology (like Kelvin-Voigt or Jeffrey) is considered, cf. [34]. Let us emphasize that this engineering model does not possess any rigorous mathematical justification in its typical engineering usage, i.e. the purely quasistatic elastic case when no viscosity is considered.

Microscopically, an additional dissipation in the Mode II may be explained by a certain plastic process both in the adhesive itself and in a narrow bulk vicinity of the delamination surface before the actual delamination starts, cf. [17, 50]. Inspired by this, another model has been devised in [40] by introducing an extra inelastic parameter which describes some *plastic-like interfacial slip* occurring possibly in the tangent direction of an interface before its debonding is activated. This interfacial plastification is not activated in Mode I, which allows for dissipating some extra energy in Mode II compared to Mode I. This model is associative in the sense that the dissipation potential depends only on rates but not states. Its rigorous analysis has been performed in [41], based on implicit discretisation in time and global minimization, using the concept of globally stable energy-conserving (so-called energetic) solutions devised by Mielke et al. [23, 24, 28, 29, 30].

It is well known, however, that energetic solutions of rate-independent problems governed by nonconvex energies (as inevitable in fracture mechanics and in particular here too) tend to nonphysically too early jumps. Instead of energy-driven and energy-conserving solution, some other concepts seem more physically relevant, like vanishing-viscosity solutions. See also the discussion about energy versus stress or global versus local minimization in mathematical literature [6, 22, 45] and in engineering [16, 21], and also the examples [5, Sect. 9] or [25, Example 7.1]. In general, all reasonable solutions fall into so-called local solutions, invented [49], cf. also [24].

Here, the undesired effect of too early jumps of globally stable energy-conserving solutions can be caused both by the influence of big energy stored in the stressed bulk (cf. the explicit example in [38]) and also by a tendency to slide to less dissipative mode of delamination (i.e. Mode I) even if the direction of the traction stress would clearly tend to a more dissipative mode (i.e. Mode II), as also indicated by numerical experiments in [41, 40].

In the mode-mixity-insensitive model (i.e. Mode I dissipates equally as Mode II), it has been observed in [42] that the local solutions obtained by semi-implicit time discretisation nicely coincides numerically with the vanishing-viscosity solutions in all investigated examples; of course, the energy conservation has been lost in such local but non-energetic solutions. Mathematical justification of the semi-implicit time discretisation for the quasistatic rate-independent problem was not scrutinized in [42], however.

The goal of this article is to devise a physically relevant model (together with a corresponding solution concept) for quasistatic mode-mixity-sensitive delamination together with an efficient numerical strategy. In Section 2, we briefly present the model devised in [40, 41] and in Section 3 we define its solution using the concept of local solutions from [24, 49]. Then, in Section 4, we devise a suitable semi-implicit time discretisation and show its unconditional stability in the sense that a-priori estimates can be proved, and then prove convergence toward the local solutions of the continuous problem. Eventually, in Section 5, we briefly present the fully discretised model and outline its unconditional convergence if the time and space discretisation refines, and present computational simulations documenting this convergence as well as physical relevancy of the model and its solution. Let us emphasize that, in particular, it is for the first time when the mode-mixity-sensitive delamination model and its solution pursuing the maximum-dissipation principle and, due to Remark 1 below, the stress-driven solution concept in purely inviscid quasistatic situation

is analyzed as far as the convergence concerns and, on a fixed discretisation, the approximate solutions can efficiently be calculated non-iteratively at each time level by using a linear-quadratic programming algorithms.

## 2. QUASISTATIC MODE-MIXITY-SENSITIVE DELAMINATION MODEL

We will consider the evolution on a fixed finite time interval  $[0, T]$  governed by a *stored energy* functional  $\mathcal{E} = \mathcal{E}(t, u, z)$  and a *dissipated energy* functional  $\mathcal{R} = \mathcal{R}(\dot{z})$  with the displacement field  $u$  and an “inelastic” parameter field  $z$  composed here from delamination and interface-plasticity parameters.

The *delamination* (or *interfacial damage*) parameter is related to fraction of adhesive bonds which are not broken. The *interface-plasticity* parameter is motivated by the idea that, microscopically, the additional dissipation in Mode II may be explained by a certain plastic processes both in the adhesive itself and in a narrow bulk vicinity of the delaminating surface before the actual delamination starts [50], or by some rough structure of the interface [8]. In a certain idealization, these plastic processes are more relevant in Mode II while do not manifest themselves significantly in Mode I if the plastic strain is considered incompressible, i.e. trace-free.

Further, we use the notation for the time derivative  $\dot{z} := \frac{dz}{dt}$ . Specification of these energy functionals will be given later. The general form of inclusions governing the rate-independent evolution scrutinized in this article is the following system of *doubly nonlinear* degenerate abstract *static/evolution inclusions*, referred sometimes as Biot’s equations generalizing the original work [3, 4]:

$$\partial_u \mathcal{E}(t, u, z) \ni 0 \quad \text{and} \quad \partial \mathcal{R}(\dot{z}) + \partial_z \mathcal{E}(t, u, z) \ni 0, \quad (1)$$

where the symbol “ $\partial$ ” refers to a (partial) subdifferential, relying on that  $\mathcal{R}(\cdot)$ ,  $\mathcal{E}(t, \cdot, z)$ , and  $\mathcal{E}(t, u, \cdot)$  are convex functionals; the latter inclusion in (1) thus contains the sum of two sets.

First we present in detail a plastic-type model with kinematic-type hardening (like e.g. in [11, 44]) for the above described delamination problem, devised, analyzed, and tested numerically in [41, 40]. The philosophy of the associative model is to consider, besides some interface damage process described by a variable  $\zeta$ , another inelastic process on the delaminating surface  $\Gamma_c$  which would be activated rather in fracture Mode II than in Mode I, and thus more energy would be dissipated in Mode II than in Mode I. This additional inelastic process involves an additional dissipative variable  $\pi$  having the meaning of the plastic-like *tangential slip* on  $\Gamma_c$ ; this variable defined on  $\Gamma_c$  is a  $(d-1)$ -dimensional vector. We will use a gradient theory for some of the internal parameters used also, e.g., in [9, Chap.14] or [41, 40]. In contrast to [41, 40], we consider here the gradient of  $\pi$  instead of  $\zeta$  because now we need strong convergence of all convex nonlinear terms, which does not seem easy for a term like  $|\nabla_s \zeta|^r$  if  $\partial \mathcal{R}$  is not bounded, as it is the case here because no re-bonding is considered, i.e. only  $\dot{\zeta} \leq 0$  is allowed, cf. (4b).

The relation to (1) is that  $z = (\zeta, \pi)$  and, confining ourselves to  $\mathcal{R}(\dot{z}, \dot{\pi}) = \mathcal{R}_0(\dot{\zeta}) + \mathcal{R}_1(\dot{\pi})$ , the system (1) takes the form

$$\partial_u \mathcal{E}(t, u, \zeta, \pi) \ni 0, \quad (\text{force equilibrium, Signorini contact}) \quad (2a)$$

$$\partial \mathcal{R}_0(\dot{\zeta}) + \partial_\zeta \mathcal{E}(t, u, \zeta, \pi) \ni 0, \quad (\text{a flow rule for interfacial damage}) \quad (2b)$$

$$\partial \mathcal{R}_1(\dot{\pi}) + \partial_\pi \mathcal{E}(t, u, \zeta, \pi) \ni 0. \quad (\text{a flow rule for interfacial plasticity}) \quad (2c)$$

To formulate the model, we consider two bounded Lipschitz domains  $\Omega_1, \Omega_2 \subset \mathbb{R}^d$  ( $d = 2, 3$ ) with a common contact boundary  $\Gamma_c = \partial\Omega_1 \cap \partial\Omega_2$ ; of course, the generalization for more than 2 bodies in contact is straightforward. Occasionally, we use the notation  $\Omega = \Omega_1 \cup \Gamma_c \cup \Omega_2$ . The contact boundary  $\Gamma_c$  may undergo delamination. We assume that the rest of the outer boundary  $\partial\Omega$  is (up to  $d-2$ -dimensional zero-measure set) the union of two disjoint open subsets  $\Gamma_D$  and  $\Gamma_N$  where the Dirichlet and the Neumann boundary conditions

will be prescribed, respectively. To ensure coercivity of the problem even after a possible complete delamination, we assume

$$\partial\Omega_1 \cap \Gamma_D \neq \emptyset \quad \& \quad \partial\Omega_2 \cap \Gamma_D \neq \emptyset. \quad (3)$$

On the Dirichlet part  $\Gamma_D$  of the boundary, we impose a time-dependent displacement  $w_D(t)$  and, on the boundary  $\Gamma_N$ , we impose a time-dependent traction  $f(t)$ .

The introduced associative delamination model is determined by the stored-energy functional

$$\mathcal{E}(t, u, \zeta, \pi) := \begin{cases} \int_{\Omega \setminus \Gamma_C} \frac{1}{2} \mathbb{C} e(u) : e(u) \, dx - \int_{\Gamma_N} f(t) \cdot u \, dS \\ \quad + \int_{\Gamma_C} \left( \zeta \left( \frac{\kappa_N}{2} [u]_N^2 + \frac{\kappa_T}{2} |[u]_T - \mathbb{T}\pi|^2 \right) \right. \\ \quad \left. + \frac{\kappa_H}{2} |\pi|^2 + \frac{\kappa_G}{2} |\nabla_s \pi|^2 \right) dS & \text{if } u = w_D(t) \text{ on } \Gamma_D, \\ \quad [u]_N \geq 0, \, 0 \leq \zeta \leq 1 \text{ on } \Gamma_C, \\ \infty & \text{elsewhere,} \end{cases} \quad (4a)$$

and by the dissipated-energy functional

$$\mathcal{R}_0(\dot{\zeta}) := \begin{cases} \int_{\Gamma_C} a_1 |\dot{\zeta}| \, dS & \text{if } \dot{\zeta} \leq 0 \text{ a.e. on } \Gamma_C, \\ \infty & \text{otherwise,} \end{cases} \quad \mathcal{R}_1(\dot{\pi}) := \int_{\Gamma_C} \sigma_{\text{yield}} |\dot{\pi}| \, dS, \quad (4b)$$

with  $\mathbb{C}$  being the elastic-moduli tensor (possibly being  $x$ -dependent and, in particular, may be different at the subdomains  $\Omega_1$  and  $\Omega_2$ ),  $e(u) = \frac{1}{2}(\nabla u)^\top + \frac{1}{2}\nabla u$  denoting the small strain tensor,  $a_1 > 0$  the prescribed phenomenological energy per unit area dissipated (= needed for complete delamination, often referred to as fracture energy or fracture toughness) in pure Mode I,  $\nabla_s$  a “surface gradient” (i.e. the tangential derivative defined as  $\nabla_s v = \nabla v - (\nabla v \cdot \nu)\nu$  for  $v$  defined around  $\Gamma_C$ ) and  $[u] = [u]_N \nu + [u]_T$  with  $[u]_N = [u] \cdot \nu$  with  $\nu$  a unit normal to  $\Gamma_C$ ; in other words,  $\nabla_s v := P(\nabla v)$  and  $[u]_T = [Pu]$  with the projector  $P = \mathbb{I} - \nu \otimes \nu$  onto a tangent space. Here we used the notation  $[u]$  for the differences of traces from both sides of  $\Gamma_C$ . Note also that  $[u]_N$  is scalar valued while  $[u]_T$  is vector valued. Alternatively, pursuing the concept of fields defined exclusively on  $\Gamma$ , we can consider  $v : \Gamma \rightarrow \mathbb{R}^d$  and extend it to a neighborhood of  $\Gamma$  and then again define  $\nabla_s v := P(\nabla v)$  which, in fact, does not depend on the particular extension. In fact, the model naturally does not depend on the chosen orientation. The value  $\infty$  in (4b) guarantees that  $\dot{\zeta} \leq 0$  during the whole evolution everywhere on  $\Gamma_C$ , i.e. the interfacial damage evolution is irreversible, we can also say uni-directional, in the sense that no re-bonding (i.e. no healing) is allowed. In (4a),  $\mathbb{T} : \Gamma_C \rightarrow \text{Lin}(\mathbb{R}^{d-1}, \mathbb{R}^d)$  is to embed  $\mathbb{R}^{d-1}$ , where  $\pi$  is valued, into  $\mathbb{R}^d$ , where  $[u]_T$  has values, so that  $[u]_T - \mathbb{T}\pi$  has a sense, and we assume that  $\mathbb{T}(x)(\mathbb{R}^{d-1})$  is the  $(d-1)$ -dimensional tangent space to  $\Gamma_C$  at  $x$  for a.a.  $x \in \Gamma_C$ . The phenomenological elastic constants  $\kappa_N$  and  $\kappa_T$  in (4a) describe the stiffnesses of linearly elastically responding adhesive in the normal and tangential directions, respectively. Typical phenomenology is that  $\kappa_N$  is greater than  $\kappa_T$ ; even, for isotropic adhesive, a condition  $\kappa_N/\kappa_T \geq 2$  has been deduced in [48], see also further references therein.

The unilateral constraint  $[u]_N \geq 0$  in (4a) guarantees infinitesimal nonpenetration before and after delamination (the so-called Signorini contact) and impossibility of delamination by pure compression. To produce desired effects, the model should work with parameters satisfying

$$\frac{1}{2} \kappa_T a_1 < \sigma_{\text{yield}}^2 \leq 2 \kappa_T a_1. \quad (5)$$

More specifically, the upper bound of the yield stress is necessary for making possible to initiate plastic slip before the total interface damage, while the lower bound is required to

avoid plastic slip evolution after complete debonding when  $\zeta = 0$ . Then, one can see that the overall dissipated energy in Mode II, denoted by  $a_{\text{II}}$ , is

$$a_{\text{II}} = a_{\text{I}} + \frac{2\kappa_{\text{T}}a_{\text{I}} - \sigma_{\text{yield}}^2}{2\kappa_{\text{H}}} \quad (6)$$

cf. [41, 40] where, however, the contribution of the hardening after complete delamination was ignored. This hardening energy, although being a part of the stored energy, cannot be gained back (assuming (2.5)) and thus is effectively dissipated for ever after the delamination in Mode II is completed. For example, for  $\sigma_{\text{yield}} = \sqrt{\kappa_{\text{T}}a_{\text{I}}}$ , (5) is satisfied and  $a_{\text{II}}/a_{\text{I}} = 1 + \kappa_{\text{T}}/(2\kappa_{\text{H}})$ . In particular, by choosing  $\kappa_{\text{H}} > 0$  small, this model can handle arbitrarily large ratio  $a_{\text{II}}/a_{\text{I}}$ ; let us emphasize that in engineering situations, this ratio is often over 10.

The typical occurrence of jumps of solutions needs a careful definition relying on the time derivative  $\mathcal{E}'_t(\cdot, u, \zeta, \pi)$  for  $(u, \zeta, \pi)$  fixed, cf. the last term in (9d). This obviously requires  $w_{\text{D}}$  in (4a) constant in time to avoid the situation that, for  $u$  fixed such that  $u|_{\Gamma_{\text{D}}} = w_{\text{D}}(t_0)$ , the value  $\mathcal{E}(t, u, \zeta, \pi)$  is finite for  $t = t_0$  while it equals  $\infty$  for  $t \neq t_0$  and thus  $\mathcal{E}'_t(\cdot, u, \zeta, \pi)$  cannot exist at  $t = t_0$ . In a general case  $\dot{w}_{\text{D}} \neq 0$ , we make a substitution of  $u + u_{\text{D}}(t)$  with  $u_{\text{D}}(t)$  being a suitable extension of  $w_{\text{D}}(t)$ . Then, up to the time-dependent constant  $\int_{\Omega \setminus \Gamma_{\text{C}}} \frac{1}{2} \mathbb{C}e(u_{\text{D}}(t)) : e(u_{\text{D}}(t)) \, dx - \int_{\Gamma_{\text{N}}} f(t) \cdot u_{\text{D}}(t) \, dS$ , (4a) is to be replaced by

$$\mathcal{E}(t, u, \zeta, \pi) := \begin{cases} \int_{\Omega \setminus \Gamma_{\text{C}}} \frac{1}{2} \mathbb{C}e(u) : e(u) \, dx - \langle f_1(t), u \rangle \\ \quad + \int_{\Gamma_{\text{C}}} \left( \zeta \left( \frac{\kappa_{\text{N}}}{2} [u]_{\text{N}}^2 + \frac{\kappa_{\text{T}}}{2} |[u]_{\text{T}} - \mathbb{T}\pi|^2 \right) \right. \\ \quad \left. + \frac{\kappa_{\text{H}}}{2} |\pi|^2 + \frac{\kappa_{\text{G}}}{2} |\nabla_{\text{s}} \pi|^2 \right) dS & \text{if } u = 0 \text{ on } \Gamma_{\text{D}} \text{ and} \\ & \llbracket u \rrbracket_{\text{N}} \geq 0, \, 0 \leq \zeta \leq 1 \text{ on } \Gamma_{\text{C}}, \\ \infty & \text{elsewhere,} \end{cases} \quad (7a)$$

$$\text{with } \langle f_1(t), v \rangle = \int_{\Gamma_{\text{N}}} f(t) \cdot v \, dS - \int_{\Omega \setminus \Gamma_{\text{C}}} \mathbb{C}e(u_{\text{D}}(t)) : e(v) \, dx. \quad (7b)$$

In fact, we have assumed that  $\Gamma_{\text{D}}$  is far from  $\Gamma_{\text{C}}$  so that we can have  $u_{\text{D}}(t)|_{\Gamma_{\text{C}}} = 0$  not to affect the integral over  $\Gamma_{\text{C}}$  in (4a) by the shift  $u \mapsto u + u_{\text{D}}(t)$ .

An alternative way is to avoid this transformation by considering only the trace of  $u$  on  $\Gamma_{\text{C}}$  as the state variable instead of  $u$ . This is possible by using the boundary-integral-equation (BIE) method which evaluates the bulk integral and eliminates the constraint  $u|_{\Gamma_{\text{D}}} = w_{\text{D}}(t_0)$  in (4a) by solving the boundary-value problem governed by minimization of this integral under the condition that  $u$  is prescribed on  $\Gamma_{\text{D}} \cup \Gamma_{\text{C}}$ . After spatial discretisation, BIE becomes the boundary-element method, which is in fact how we implement the problem in Section 5 below, although the analysis is performed on the more conventional base of the transformed functional (7).

We will consider an initial-value problem for the system (2) by prescribing

$$u(0) = u_0, \quad \zeta(0) = \zeta_0, \quad \pi(0) = \pi_0. \quad (8)$$

### 3. LOCAL SOLUTIONS

We will also abbreviate the time interval  $I = [0, T]$  with  $T > 0$  a fixed time horizon, and  $\Sigma_{\text{C}} = I \times \Gamma_{\text{C}}$ .

We will use the standard notation  $W^{1,p}(\Omega)$  for the Sobolev space of functions having the gradient in the Lebesgue space  $L^p(\Omega; \mathbb{R}^d)$ . If valued in  $\mathbb{R}^n$  with  $n \geq 2$ , we will write  $W^{1,p}(\Omega; \mathbb{R}^n)$ , and furthermore we use the shorthand notation  $H^1(\Omega; \mathbb{R}^n) = W^{1,2}(\Omega; \mathbb{R}^n)$ . Similarly, we will use Lebesgue and Sobolev space on the  $(d-1)$ -dimensional manifold  $\Gamma_{\text{C}}$ , assumed Lipschitz so that a local rectification for defining the surface gradient  $\nabla_{\text{s}}$  can be performed a.e. on  $\Gamma_{\text{C}}$ . We also use the notation of “ $\cdot$ ” and “ $:$ ” for a scalar product of

vectors and 2nd-order tensors, respectively. For a Banach space  $X$ ,  $L^p(I; X)$  will denote the Bochner space of  $X$ -valued Bochner measurable functions  $u : I \rightarrow X$  with its norm  $\|u(\cdot)\|$  in  $L^p(I)$ , here  $\|\cdot\|$  stands for the norm in  $X$ . Further,  $W^{1,p}(I; X)$  denotes the Banach space of mappings  $u : I \rightarrow X$  whose distributional time derivative is in  $L^p(I; X)$ , while  $BV(I; X)$  will denote the space of mappings  $u : I \rightarrow X$  with a bounded variations, i.e.  $\sup_{0 \leq t_0 < t_1 < \dots < t_{n-1} < t_n \leq T} \sum_{i=1}^n \|u(t_i) - u(t_{i-1})\| < \infty$  where the supremum is taken over all finite partitions of the interval  $I = [0, T]$ . By  $B(I; X)$  we denote the space of bounded measurable (everywhere defined) mapping  $I \rightarrow X$ .

The concept of local solutions has been introduced for a special crack problem in [49] and independently also in [45], and further generally investigated in [24]. Here, we additionally combine it with the concept of semi-stability as invented in [35]. We adapt the general definition directly to our specific problem, which will lead to two semi-stability conditions for  $\zeta$  and  $\pi$ , respectively:

**Definition 1** (Local solutions). We call a measurable mapping  $(u, \zeta, \pi) : I \rightarrow H^1(\Omega \setminus \Gamma_c; \mathbb{R}^d) \times L^\infty(\Gamma_c) \times H^1(\Gamma_c; \mathbb{R}^{d-1})$  a local solution to the delamination problem (2)–(8) if the initial conditions (8) are satisfied, if  $\llbracket u \rrbracket_N \geq 0$  on  $\Sigma_c$  and, for some  $J \subset I$  at most countable (containing time instances where the solution may possibly jump), it holds that:

$$\begin{aligned} \forall t \in I \setminus J \quad \forall v \in H^1(\Omega \setminus \Gamma_c; \mathbb{R}^d), \quad \llbracket v \rrbracket_N \geq 0 : \\ \langle \partial_u \mathcal{E}(t, u(t), \zeta(t), \pi(t)), v - u(t) \rangle \geq 0, \end{aligned} \quad (9a)$$

$$\begin{aligned} \forall t \in I \quad \forall \tilde{\zeta} \in L^\infty(\Gamma_c), \quad 0 \leq \tilde{\zeta} \leq \zeta(t) \text{ a.e. on } \Gamma_c : \\ \mathcal{E}(t, u(t), \zeta(t), \pi(t)) \leq \mathcal{E}(t, u(t), \tilde{\zeta}, \pi(t)) + \mathcal{R}_0(\tilde{\zeta} - \zeta(t)), \end{aligned} \quad (9b)$$

$$\begin{aligned} \forall t \in I \setminus J \quad \forall \tilde{\pi} \in H^1(\Gamma_c; \mathbb{R}^{d-1}) : \\ \mathcal{E}(t, u(t), \zeta(t), \pi(t)) \leq \mathcal{E}(t, u(t), \zeta(t), \tilde{\pi}) + \mathcal{R}_1(\tilde{\pi} - \pi(t)), \end{aligned} \quad (9c)$$

$$\begin{aligned} \forall 0 \leq t_1 \leq t_2 \leq T : \quad \mathcal{E}(t_2, u(t_2), \zeta(t_2), \pi(t_2)) + \text{Diss}_{\mathcal{R}_1}(\pi; [t_1, t_2]) \\ + \int_{\Gamma_c} a_1(\zeta(t_1) - \zeta(t_2)) \, dS \leq \mathcal{E}(t_1, u(t_1), \zeta(t_1), \pi(t_1)) - \int_{t_1}^{t_2} \langle \dot{f}_1, u \rangle \, dt \end{aligned} \quad (9d)$$

where  $f_1$  is from (7b) and  $\text{Diss}_{\mathcal{R}_1}(\pi; [r, s]) := \sup \sum_{j=1}^N \sigma_{\text{yield}} |\pi(t_{j-1}) - \pi(t_j)|$  with the supremum taken over all finite partitions  $r \leq t_0 < t_1 < \dots < t_{N-1} < t_N \leq s$ .

Let us comment the above definition briefly. Obviously, (2a) means precisely (9a), which more in detail here means that  $\int_{\Omega \setminus \Gamma_c} \mathbb{C}e(u(t)) : e(v - u(t)) \, dx + \int_{\Gamma_c} \zeta(t) (\kappa_N \llbracket u(t) \rrbracket_N, \kappa_T \llbracket u(t) \rrbracket_T - \mathbb{T}\pi) \cdot \llbracket v - u(t) \rrbracket \, dS \geq \langle f_1(t), v - u(t) \rangle$  for all  $v \in H^1(\Omega \setminus \Gamma_c; \mathbb{R}^d)$  with  $\llbracket v \rrbracket_N \geq 0$ . Note that (9a) specifies also the boundary conditions for  $u$ , namely  $u = 0$  on  $\Gamma_b$  because otherwise  $\mathcal{E}(t, u, \zeta, \pi) = \infty$  would violate (9a) for  $v$  which satisfies  $v = 0$  on  $\Gamma_b$ , and also  $v^\top \mathbb{C}e(u) = f$  on  $\Gamma_N$  can be proved by standard arguments based on Green's theorem. As  $\mathcal{R}_1$  is homogeneous degree-1, always  $\partial \mathcal{R}_1(\dot{\pi}) \subset \partial \mathcal{R}_1(0)$  and thus (2c) implies  $\partial \mathcal{R}_1(0) + \partial_\pi \mathcal{E}(u, \zeta, \pi) \ni 0$ . From the convexity of  $\mathcal{R}_1$  when taking into account that  $\mathcal{R}_1(0) = 0$ , the latter inclusion is equivalent to  $\mathcal{R}_1(v) + \langle \partial_\pi \mathcal{E}(u(t), \zeta(t), \pi(t)), v \rangle \geq 0$  for any  $v \in H^1(\Gamma_c; \mathbb{R}^{d-1})$ . Substituting  $v = \tilde{\zeta} - \zeta(t)$  and using the convexity of  $\mathcal{E}(t, u, \zeta, \cdot)$ , we obtain the *semi-stability* (9c) of  $\pi$  at time  $t$ . Analogously, we obtain also (9b) from (2b). Eventually, (9d) is the (im)balance of the mechanical energy with the last term representing a “complementary” work of external forces arising from the usual work by a by-part integration in time. This generalizes the standard definition of the *weak solution* to (2) to the case when  $\mathcal{E}(t, \cdot, \cdot, \cdot)$  is not smooth, cf. [39] for details.

To be more precise, the concept of local solutions as used in [24, 49] requires  $J$  only to have a zero Lebesgue measure and also (9b) is valid only for a.a.  $t$ . On the other hand, conventional weak solutions allow even (9d) holding only for a.a.  $t_1$  and  $t_2$ . Later, our approximation method will provide convergence to this slightly stronger local solutions, which motivates us to have tailored Definition 1 straight to our results.



Actually, local solutions form essentially the largest reasonable class of solutions for (1), coinciding (in the above mentioned weaker form) with the conventional weak solutions, cf. [39]. It includes the mentioned energetic solutions [23, 29], the vanishing-viscosity solutions, the balanced-viscosity (so-called BV) solutions, parametrized solutions, etc.; cf. [24, 27] for a survey, and also stress-driven-like solutions obeying maximum-dissipation principle in some sense. The energetic solution has often tendency to rupture unphysically early and rather in the less dissipative Mode I even if there should be rather Mode II expected; cf. [51] for a comparison on several computational experiments. The approximation method we will use in this article leads rather to the stress-driven option, cf. Remarks 1 and 3 below.

Anyhow, let us mention that, in [41], existence of the globally stable energy-conserving local solutions of this model has been proved under the following assumptions:

$$\mathbb{C}^{(i)} \text{ positive definite, symmetric, } \kappa_G, \kappa_H > 0, \kappa_T, \kappa_N \geq 0, a_1, \sigma_{\text{yield}} > 0, \quad (10a)$$

$$w_D \in W^{1,1}(0, T; W^{1/2,2}(\Gamma_D; \mathbb{R}^d)), \quad (10b)$$

$$f \in W^{1,1}(0, T; L^p(\Gamma_N; \mathbb{R}^d)) \quad \text{with} \quad p \begin{cases} > 1 & \text{for } d = 2, \\ = 2-2/d & \text{for } d \geq 3 \end{cases} \quad (10c)$$

$$(u_0, \zeta_0, \pi_0) \in H^1(\Omega \setminus \Gamma_C; \mathbb{R}^d) \times L^\infty(\Gamma_C) \times H^1(\Gamma_C; \mathbb{R}^{d-1}), \quad (10d)$$

$$\forall (\tilde{u}, \tilde{\zeta}, \tilde{\pi}) : \mathcal{E}(0, u_0, \zeta_0, \pi_0) \leq \mathcal{E}(0, \tilde{u}, \tilde{\zeta}, \tilde{\pi}) + \mathcal{R}(\tilde{\zeta} - \zeta, \tilde{\pi} - \pi). \quad (10e)$$

The last condition, called stability at  $t = 0$ , is needed to ensure energy conservation and will not be needed for general local solutions. The qualification (10b) allows for an extension  $u_D$  of  $w_D$  which belongs to  $W^{1,1}(0, T; H^1(\Omega; \mathbb{R}^d))$ ; in what follows, we will consider some extension with this property.

**Remark 1** (Maximum-dissipation principle). The degree-1 homogeneity of  $\mathcal{R}_0$  and  $\mathcal{R}_1$  defined in (4b) allows for further interpretation of the flow rules (2b) and (2c). Using maximal-monotonicity of the subdifferential, (2c) means just that  $\langle \tilde{f} - f, v - \dot{\pi} \rangle \geq 0$  for any  $v$  and any  $\tilde{f} \in \partial \mathcal{R}_1(v)$  with the available driving force  $f \in -\partial_\pi \mathcal{E}(t, u, \zeta, \pi)$ ; the adjective “available” becomes sensible especially if  $\partial_\pi \mathcal{E}(t, u, \zeta, \pi)$  is set-valued because not all available  $f$ ’s are compatible with  $f \in \partial \mathcal{R}_1(\dot{\pi})$  and can be realized during evolution. In particular, for  $v = 0$ , defining the convex set  $K_1 := \partial \mathcal{R}_1(0)$ , one obtains

$$\langle \dot{f}(t), \dot{\pi}(t) \rangle = \max_{\tilde{f} \in K_1} \langle \tilde{f}, \dot{\pi}(t) \rangle \quad \text{with some } f(t) \in -\partial_\pi \mathcal{E}(t, u(t), \zeta(t), \pi(t)). \quad (11a)$$

To derive it, we have used that  $f \in \partial \mathcal{R}_1(\dot{\pi}) \subset \partial \mathcal{R}_1(0) = K_1$  thanks to the degree-0 homogeneity of  $\partial \mathcal{R}_1$ , so that always  $\langle f, \dot{\pi} \rangle \leq \max_{\tilde{f} \in K_1} \langle \tilde{f}, \dot{\pi} \rangle$ . The identity (11a) says that the dissipation due to the driving force  $f$  is maximal provided that the order-parameter rate  $\dot{\zeta}$  is kept fixed, while the vector of possible driving forces  $\tilde{f}$  varies freely over all admissible driving force from  $K_1$ . This just resembles the so-called Hill’s *maximum-dissipation principle* articulated just for plasticity in [13]. Also it says that the rates are orthogonal to the “elastic domain”  $K_1$ , known as an orthogonality principle [52] generalizing Onsager’s principle [31]. See also [10, 18, 33, 53]. Actually, R. Hill [13] used it for a situation where  $\mathcal{E}(t, \cdot)$  is convex while, in a general nonconvex case as also here, it holds only along absolutely continuous paths (i.e. in stick or slip regimes) which are sufficiently regular in the sense  $\dot{\pi}$  is valued not only in  $L^1(\Gamma_C; \mathbb{R}^{d-1})$  but also in  $H^1(\Gamma_C; \mathbb{R}^{d-1})^*$  but it does certainly not need to hold during jumps. Analogously it holds also for  $\zeta$ , defining  $K_0 := \partial \mathcal{R}_0(0)$ , i.e.

$$\langle \dot{g}(t), \dot{\zeta}(t) \rangle = \max_{\tilde{g} \in K_0} \langle \tilde{g}, \dot{\zeta}(t) \rangle \quad \text{with some } g(t) \in -\partial_\zeta \mathcal{E}(t, u(t), \zeta(t), \pi(t)). \quad (11b)$$

As  $\mathcal{E}(t, u, \zeta, \cdot)$  is smooth, the maximum-dissipation relation (11a) written in the form

$$\langle -\mathcal{E}'_\pi(t, u(t), \zeta(t), \pi(t)), \dot{\pi}(t) \rangle = \max_{K_1} \langle K_1, \dot{\pi}(t) \rangle = \mathcal{R}_1(\dot{\pi}(t))$$

summed with the semistability (9c) which can be written in the form

$$\mathcal{R}_1(\bar{\pi}) + \langle \mathcal{E}'_\pi(t, u(t), \zeta(t), \pi(t)), \bar{\pi} \rangle \geq 0$$

thanks to the convexity of  $\mathcal{E}(t, u, \zeta, \cdot)$  yields

$$\mathcal{R}_1(\bar{\pi}) + \langle \mathcal{E}'_\pi(t, u(t), \zeta(t), \pi(t)), \bar{\pi} - \dot{\pi}(t) \rangle \geq \mathcal{R}_1(\dot{\pi}(t)) \quad (12)$$

for any  $\bar{\pi}$ , which just means that  $\mathfrak{f}(t) = -\mathcal{E}'_\pi(t, u(t), \zeta(t), \pi(t)) \in \partial \mathcal{R}_1(\dot{\pi}(t))$ . This exactly means that the evolution of  $\pi$  is *governed by a thermodynamical driving force*  $\mathfrak{f}$  (we say that it is “stress-driven”) and it reveals the role of the maximum-dissipation principle in combination with semistability. Using the convexity of  $\mathcal{E}(t, u, \cdot, \pi)$ , a similar argument can be applied for (11b) in combination with semistability (9b) even if  $\mathcal{E}(t, u, \cdot, \pi)$  is not smooth.

**Remark 2** (Integrated maximum-dissipation principle). Let us emphasize that, in general,  $\dot{\zeta}$  and  $\dot{\pi}$  are measures possibly having singular parts concentrated at rupture times where the solution and also the driving forces need not be continuous. Even if  $\dot{\zeta}$  and  $\dot{\pi}$  are absolutely continuous, in our infinite-dimensional case the driving forces need not be in duality with them, as already mentioned in Remark 1. So (11) is analytically not justified in any sense. For this reason, an *Integrated* version of the *Maximum-Dissipation Principle* (IMDP) was devised in [39] for a bit simpler case involving only one maximum-dissipation relation. Realizing that  $\max_{\bar{\mathfrak{f}} \in K_1} \langle \bar{\mathfrak{f}}, \dot{\pi} \rangle = \mathcal{R}_1(\dot{\pi})$  and similarly  $\max_{\bar{\mathfrak{g}} \in K_0} \langle \bar{\mathfrak{g}}, \dot{\zeta} \rangle = \mathcal{R}_0(\dot{\zeta})$ , the integrated version of (11) reads here as:

$$\int_{t_1}^{t_2} \mathfrak{f}(t) d\pi(t) = \int_{t_1}^{t_2} \mathcal{R}_1(\dot{\pi}) dt \quad \text{with some } \mathfrak{f}(t) \in -\partial_\pi \mathcal{E}(t, u(t), \zeta(t), \pi(t)), \quad (13a)$$

$$\int_{t_1}^{t_2} \mathfrak{g}(t) d\zeta(t) = \int_{t_1}^{t_2} \mathcal{R}_0(\dot{\zeta}) dt \quad \text{with some } \mathfrak{g}(t) \in -\partial_\zeta \mathcal{E}(t, u(t), \zeta(t), \pi(t)) \quad (13b)$$

to be valid for any  $0 \leq t_1 < t_2 \leq T$ . This definition is inevitably a bit technical and, without sliding into too much details, let us only mention that the left-hand-side integrals in (13) are the so-called *lower Riemann-Stieltjes integrals* defined by suprema of lower Darboux sums, i.e. in the case (13a) as

$$\int_r^s \mathfrak{f}(t) d\pi(t) := \sup_{\substack{N \in \mathbb{N} \\ r=t_0 < t_1 < \dots < t_{N-1} < t_N=s}} \sum_{j=1}^N \inf_{t \in [t_{j-1}, t_j]} \langle \mathfrak{f}(t), \pi(t_j) - \pi(t_{j-1}) \rangle,$$

while the right-hand-side integrals are just the integrals of measures and equal to  $\text{Diss}_{\mathcal{R}_1}(\pi; [t_1, t_2])$  and  $\text{Diss}_{\mathcal{R}_0}(\zeta; [t_1, t_2])$ , respectively. The IMDP (13) is satisfied on any interval  $[t_1, t_2]$  where the solution is absolutely continuous with sufficiently regular time derivatives; then the integrals in (13) are the conventional Lebesgue integrals, in particular the left-hand sides in (13) are  $\int_{t_1}^{t_2} \langle \mathfrak{f}(t), \dot{\pi}(t) \rangle dt$  and  $\int_{t_1}^{t_2} \langle \mathfrak{g}(t), \dot{\zeta}(t) \rangle dt$ , respectively. The particular importance of IMDP is especially at jumps, i.e. at times when abrupt delamination possibly happens. It is shown in [27, 39] on various finite-dimensional examples of “damageable springs” that this IMDP can identify too early rupturing local solutions when the driving force is obviously unphysically low (which occurs quite typically in particular within the energetic solutions of systems governed by nonconvex potentials like here) and its satisfaction for left-continuous local solutions indicates that the evolution is stress driven, as explained in Remark 1. On the other hand, it does not need to be satisfied even in physically well justified stress-driven local solutions. For example, it happens if two springs with different fracture toughness organized in parallel rupture at the same time (although even in this situation our algorithm (14) below will give a correct approximate solution). Therefore, even the IMDP (13) may serve only as a sufficient aposteriori condition whose satisfaction verifies the obtained local solution as a physically relevant in the sense that it is stress driven but its dissatisfaction does not mean anything. Moreover, we will rely rather on some approximation of IMDP, as described in Remark 3 below.



#### 4. SEMI-IMPLICIT TIME DISCRETISATION, ITS STABILITY AND CONVERGENCE

To prove existence of the physically relevant solution, we use a constructive method relying on time discretisation and the weak compactness of level sets of the minimization problems arising at each time level. When further discretised in space, it will later in Sect. 5 yield a computer implementable efficient algorithm.

For the mentioned time discretisation, we use an equidistant partition of the time interval  $I = [0, T]$  with a time step  $\tau > 0$ , assuming  $T/\tau \in \mathbb{N}$ , and denote  $\{u_\tau^k\}_{k=0}^{T/\tau}$  an approximation of the desired values  $u(k\tau)$ , and similarly  $\zeta_\tau^k$  is to approximate  $\zeta(k\tau)$ , etc.

We use a decoupled semi-implicit time discretisation with the fractional steps based on the splitting of the state variables governed by the separately-convex character of  $\mathcal{E}(t, \cdot, \cdot, \cdot)$ . This will make the numerics considerably easier than any other splitting and simultaneously may lead to a physically relevant solutions governed rather by stresses (if the maximum-dissipation principle holds at least approximately in the sense of Remark 3 below) than by energies and will prevent too-early debonding, as already announced in Section 1. More specifically, exploiting the convexity of both  $\mathcal{E}(t, \cdot, \zeta, \cdot)$  and  $\mathcal{E}(t, u, \cdot, \pi)$ , this splitting will be considered as  $(u, \pi)$  and  $\zeta$ . This yields alternating convex minimization. Thus, for  $(\zeta_\tau^{k-1}, \pi_\tau^{k-1})$  given, we obtain two minimization problems

$$\left. \begin{array}{ll} \text{minimize} & \mathcal{E}(k\tau, u, \zeta_\tau^{k-1}, \pi) + \mathcal{R}_1(\pi - \pi_\tau^{k-1}) \\ \text{subject to} & (u, \pi) \in H^1(\Omega \setminus \Gamma_C; \mathbb{R}^d) \times H^1(\Gamma_C; \mathbb{R}^{d-1}), \end{array} \right\} \quad (14a)$$

and, denoting the unique solution as  $(u_\tau^k, \pi_\tau^k)$ ,

$$\left. \begin{array}{ll} \text{minimize} & \mathcal{E}(k\tau, u_\tau^k, \zeta, \pi_\tau^k) + \mathcal{R}_0(\zeta - \zeta_\tau^{k-1}) \\ \text{subject to} & \zeta \in L^\infty(\Gamma_C), \quad 0 \leq \zeta \leq 1, \end{array} \right\} \quad (14b)$$

and denote its (possibly not unique) solution by  $\zeta_\tau^k$ .

Existence of the discrete solutions  $(u_\tau^k, \zeta_\tau^k, \pi_\tau^k)$  is straightforward by the mentioned compactness arguments. Rather, it is important that both problems (14) have the linear-quadratic structure, the former one after applying the Mosco-type transformation, cf. [37, Lemma 4]. This obviously facilitates their numerical treatment; cf. Section 5 below.

We define the piecewise-constant interpolants

$$\left. \begin{array}{ll} \bar{u}_\tau(t) = u_\tau^k & \& \underline{u}_\tau(t) = u_\tau^{k-1}, \\ \bar{\zeta}_\tau(t) = \zeta_\tau^k & \& \underline{\zeta}_\tau(t) = \zeta_\tau^{k-1}, \\ \bar{\pi}_\tau(t) = \pi_\tau^k & \& \underline{\pi}_\tau(t) = \pi_\tau^{k-1}, \\ \bar{\mathcal{E}}_\tau(t, u, \zeta, \pi) = \mathcal{E}(k\tau, u, \zeta, \pi) \end{array} \right\} \text{ for } (k-1)\tau < t \leq k\tau. \quad (15)$$

Later in Remark 3, we will use also the piecewise affine interpolants

$$\left. \begin{array}{l} \zeta_\tau(t) = \frac{t-(k-1)\tau}{\tau} \zeta_\tau^k + \frac{k\tau-t}{\tau} \zeta_\tau^{k-1}, \\ \pi_\tau(t) = \frac{t-(k-1)\tau}{\tau} \pi_\tau^k + \frac{k\tau-t}{\tau} \pi_\tau^{k-1} \end{array} \right\} \text{ for } (k-1)\tau < t \leq k\tau. \quad (16)$$

The important attribute of the discretisation (14) is also its numerical stability and satisfaction of a suitable discrete analog of (9), namely:

**Proposition 1** (Stability of the time discretisation). Let (10a-d) hold and, in terms of the interpolants (15),  $(\bar{u}_\tau, \bar{\zeta}_\tau, \bar{\pi}_\tau)$  be an approximate solution obtained by (14). Then, the following a-priori estimates holds

$$\|\bar{u}_\tau\|_{L^\infty(I; H^1(\Omega \setminus \Gamma_C; \mathbb{R}^d))} \leq C, \quad (17a)$$

$$\|\bar{\zeta}_\tau\|_{L^\infty(\Sigma_C) \cap BV(I; L^1(\Gamma_C))} \leq C, \quad (17b)$$

$$\|\bar{\pi}_\tau\|_{L^\infty(I; H^1(\Gamma_C; \mathbb{R}^{d-1})) \cap BV(I; L^1(\Gamma_C; \mathbb{R}^{d-1}))} \leq C. \quad (17c)$$

Moreover, the obtained approximate solution satisfies for any  $t \in I$  the variational inequality for the displacement:

$$\begin{aligned} \forall \bar{u} \in H^1(\Omega \setminus \Gamma_c; \mathbb{R}^d), \quad [\bar{u}]_N \geq 0 : \\ \langle \partial_u \mathcal{E}(t, \bar{u}_\tau(t), \underline{\zeta}_\tau(t), \bar{\pi}_\tau(t)), \bar{u} - \bar{u}_\tau(t) \rangle \geq 0, \end{aligned} \quad (18a)$$

with  $t_\tau := \min\{k\tau \geq t; k \in \mathbb{N}\}$ , two separate semi-stability conditions for  $\bar{\zeta}_\tau$  and  $\bar{\pi}_\tau$ :

$$\begin{aligned} \forall \bar{\zeta} \in L^\infty(\Gamma_c), \quad 0 \leq \bar{\zeta} \leq \bar{\zeta}_\tau(t) : \\ \mathcal{E}(t, \bar{u}_\tau(t), \bar{\zeta}_\tau(t), \bar{\pi}_\tau(t)) \leq \mathcal{E}(t, u(t), \bar{\zeta}, \bar{\pi}_\tau(t)) + \mathcal{R}_0(\bar{\zeta} - \bar{\zeta}_\tau(t)), \end{aligned} \quad (18b)$$

$$\begin{aligned} \forall \bar{\pi} \in H^1(\Gamma_c; \mathbb{R}^{d-1}) : \\ \mathcal{E}(t, \bar{u}_\tau(t), \underline{\zeta}_\tau(t), \bar{\pi}_\tau(t)) \leq \mathcal{E}(t, \bar{u}_\tau(t), \underline{\zeta}_\tau(t), \bar{\pi}) + \mathcal{R}_1(\bar{\pi} - \bar{\pi}_\tau(t)), \end{aligned} \quad (18c)$$

and the energy (im)balance:

$$\begin{aligned} \mathcal{E}(t_2, \bar{u}_\tau(t_2), \bar{\zeta}_\tau(t_2), \bar{\pi}_\tau(t_2)) + \text{Diss}_{\mathcal{R}_1}(\bar{\pi}_\tau; [t_1, t_2]) \\ + \mathcal{R}_0(\bar{\zeta}_\tau(t_2) - \bar{\zeta}_\tau(t_1)) \leq \mathcal{E}(t_1, \bar{u}_\tau(t_1), \bar{\zeta}_\tau(t_1), \bar{\pi}_\tau(t_1)) - \int_{t_1}^{t_2} \langle \dot{f}_1, \underline{u}_\tau \rangle dt, \end{aligned} \quad (18d)$$

which is to hold for all  $t \in I$  and for all  $0 \leq t_1 < t_2 \leq T$  of the form  $t_i = k_i\tau$  for some  $k_i \in \mathbb{N}$ .

*Sketch of the proof.* Writing optimality condition for (14a) in terms of  $u$ , one arrives at (18a), and comparing the value of (14a) at  $(u_\tau^k, \pi_\tau^k)$  with its value at  $(u_\tau^k, \bar{\pi})$  and using the degree-1 homogeneity of  $\mathcal{R}_1$ , one arrives at (18c).

Comparing the value of (14b) at  $\zeta_\tau^k$  with its value at  $\bar{\zeta}$  and using the degree-1 homogeneity of  $\mathcal{R}_0$ , one arrives at (18b).

In obtaining (18d), we compare the value of (14a) at the minimizer  $(u_\tau^k, \pi_\tau^k)$  with the value at  $(u_\tau^{k-1}, \pi_\tau^{k-1})$  and the value of (14b) at the minimizer  $\zeta_\tau^k$  with the value at  $\zeta_\tau^{k-1}$  and we benefit from the cancellation of the terms  $\pm \mathcal{E}(k\tau, u_\tau^k, \zeta_\tau^{k-1}, \pi_\tau^k)$ . We also use the discrete by-part integration (=summation) for the  $f_1$ -term.

Then, using (18d) for  $t_1 = 0$  and the coercivity of  $\mathcal{E}(t, \cdot, \cdot, \cdot)$  due to the assumptions (10), we obtain also the a-priori estimates (17).  $\square$

The cancellation effect in the above proof is typical in fractional-step methods, cf. e.g. [36, Remark 8.25] and for specific usage in fracture mechanics also [15]. Further, note that (18) is of a similar form as (9) and is thus prepared to make a limit passage for  $\tau \rightarrow 0$ :

**Proposition 2** (Convergence towards local solutions). Let (10a-d) hold and  $(\bar{u}_\tau, \bar{\zeta}_\tau, \bar{\pi}_\tau)$  be an approximate solution obtained by (14). Then, considering a sequence  $\tau = \tau_n = T/n$  with  $n \rightarrow \infty$ , there exists a subsequence  $\{(\bar{u}_\tau, \bar{\zeta}_\tau, \bar{\pi}_\tau)\}_{\tau>0}$  and  $u \in B(I; H^1(\Omega \setminus \Gamma_c; \mathbb{R}^d))$  with  $[\![u]\!]_N \geq 0$  on  $\Sigma_c$  and  $\zeta \in B(I; L^\infty(\Sigma_c)) \cap BV(I; L^1(\Sigma_c))$  and  $\pi \in B(I; H^1(\Sigma_c; \mathbb{R}^{d-1})) \cap BV(I; L^1(\Sigma_c; \mathbb{R}^{d-1}))$  such that

$$\bar{u}_\tau(t) \rightarrow u(t) \quad \text{in } H^1(\Omega \setminus \Gamma_c; \mathbb{R}^d) \quad \text{for all } t \in I, \quad (19a)$$

$$\bar{\zeta}_\tau(t) \xrightarrow{*} \zeta(t) \quad \text{in } L^\infty(\Gamma_c), \quad \text{for all } t \in I, \quad (19b)$$

$$\bar{\pi}_\tau(t) \rightarrow \pi(t) \quad \text{in } H^1(\Gamma_c; \mathbb{R}^{d-1}) \quad \text{for all } t \in I. \quad (19c)$$

Moreover, any  $(u, \zeta, \pi)$  obtained by this way is a local solution to the delamination problem in the sense of Definition 1.

*Proof.* By Helly's selection principle [12], cf. also e.g. [23, 24] for a more general version and usage in rate-independent processes, we choose a subsequence and  $\zeta, \underline{\zeta} \in B(I; L^\infty(\Gamma_c)) \cap BV(I; L^1(\Gamma_c))$  and  $\pi \in B(I; H^1(\Gamma_c; \mathbb{R}^{d-1})) \cap BV(I; L^1(\Gamma_c; \mathbb{R}^{d-1}))$  so that

$$\bar{\zeta}_\tau(t) \rightharpoonup \zeta(t) \quad \& \quad \underline{\zeta}_\tau(t) \rightharpoonup \underline{\zeta}(t) \quad \text{in } L^\infty(\Gamma_c) \quad \text{for all } t \in I, \quad (20a)$$

$$\bar{\pi}_\tau(t) \rightharpoonup \pi(t) \quad \text{in } H^1(\Gamma_c; \mathbb{R}^{d-1}) \quad \text{for all } t \in I. \quad (20b)$$

Now, for a fixed  $t \in I$ , by Banach's selection principle, we select (for a moment) further subsequence so that

$$\bar{u}_\tau(t) \rightharpoonup u(t) \quad \text{in } H^1(\Omega \setminus \Gamma_C; \mathbb{R}^d). \quad (21)$$

We further use that  $\bar{u}_\tau(t)$  minimizes  $\mathcal{E}(t_\tau, \cdot, \underline{\zeta}_\tau(t), \bar{\pi}_\tau)$  with  $t_\tau := \min\{k\tau \geq t; k \in \mathbb{N}\}$ . Obviously,  $t_\tau \rightarrow t$  for  $\tau \rightarrow 0$  and, by the weak-lower-semicontinuity argument, we can easily see that  $u(t)$  minimizes the strictly convex functional  $\mathcal{E}(t, \cdot, \underline{\zeta}(t), \pi(t))$ . Thus  $u(t)$  is determined uniquely so that, in fact, we did not need to make further selection of a subsequence, and this procedure can be performed for any  $t$  by using the same subsequence already selected for (20). Also,  $u : I \rightarrow H^1(\Omega \setminus \Gamma_C; \mathbb{R}^d)$  is measurable because  $\underline{\zeta}$  and  $\pi$  are measurable, and  $\partial_u \mathcal{E}(t, u(t), \underline{\zeta}(t), \pi(t)) \ni 0$  for all  $t$ .

The key ingredient is improvement of (21) for the strong convergence of displacements: by using (18a) for  $v = u(t)$  (which is a legal test because the limit  $u(t)$  satisfies the unilateral constraint  $[[u(t)]]_N \geq 0$  on  $\Gamma_C$ ), we have

$$\begin{aligned} \int_{\Omega \setminus \Gamma_C} \mathbb{C}e(\bar{u}_\tau(t) - u(t)) : e(\bar{u}_\tau(t) - u(t)) \, dx &\leq \int_{\Omega \setminus \Gamma_C} \mathbb{C}e(\bar{u}_\tau(t) - u(t)) : e(\bar{u}_\tau(t) - u(t)) \, dx \\ &\quad + \int_{\Gamma_C} \underline{\zeta}_\tau(t) \left( \kappa_N [\bar{u}_\tau(t) - u(t)]_N^2 + \kappa_T |[\bar{u}_\tau(t) - u(t)]_T|^2 \right) dS \\ &\leq \int_{\Omega \setminus \Gamma_C} \mathbb{C}e(u(t)) : e(u(t) - \bar{u}_\tau(t)) \, dx - \langle f_1(t_\tau), \bar{u}_\tau(t) - u(t) \rangle \\ &\quad + \int_{\Gamma_C} \underline{\zeta}_\tau(t) \left( \kappa_N [u(t)]_N \cdot [u(t) - \bar{u}_\tau(t)]_N \right. \\ &\quad \left. + \kappa_T ([u(t)]_T - \mathbb{T}\bar{\pi}_\tau(t)) \cdot [u(t) - \bar{u}_\tau(t)]_T \right) dS \rightarrow 0 \end{aligned} \quad (22)$$

with again  $t_\tau := \min\{k\tau \geq t; k \in \mathbb{N}\}$ . To prove this limit in (22) for  $\tau \rightarrow 0$ , we may simply use  $[[\bar{u}_\tau(t)]] \rightharpoonup [[u]]$  in  $H^{1/2}(\Gamma_C; \mathbb{R}^d)$  so strongly in  $L^2(\Gamma_C; \mathbb{R}^d)$  and  $\underline{\zeta}_\tau(t) \xrightarrow{*} \underline{\zeta}(t)$  in  $L^\infty(\Gamma_C)$  so that  $\underline{\zeta}_\tau(t) [[\bar{u}_\tau(t)]] \rightharpoonup \underline{\zeta}(t) [[u(t)]]$  in  $L^2(\Gamma_C; \mathbb{R}^d)$ . Due to the bound  $\|\bar{\pi}_\tau(t)\|_{H^1(\Gamma_C; \mathbb{R}^{d-1})}$  and the compact embedding  $H^1(\Gamma_C) \Subset L^2(\Gamma_C)$ , also  $\bar{\pi}_\tau(t) \rightarrow \pi(t)$  in  $L^2(\Gamma_C; \mathbb{R}^{d-1})$  and thus  $\underline{\zeta}_\tau(t) \mathbb{T}\bar{\pi}_\tau(t) \rightharpoonup \underline{\zeta}(t) \mathbb{T}\pi(t)$  in  $L^2(\Gamma_C; \mathbb{R}^{d-1})$ . Then the convergence in (22) is trivial. We then obtain the strong convergence (19a).

For the strong convergence (19c), we use the information from the discrete flow-rule for  $\pi$  obtained as an optimality condition for (14a) with respect to  $\pi$ , written as

$$\bar{f}_\tau + \underline{\zeta}_\tau \kappa_T ([\bar{u}_\tau]_T - \mathbb{T}\bar{\pi}_\tau) + \kappa_H \bar{\pi}_\tau + \kappa_G \operatorname{div}_s \nabla_s \bar{\pi}_\tau = 0 \quad \text{with } \bar{f}_\tau \in N_{B_{\sigma_{\text{yield}}}}(\bar{\pi}_\tau) \quad (23)$$

with  $N_{B_{\sigma_{\text{yield}}}}$  denoting the set-valued mapping  $\mathbb{R}^{d-1} \rightrightarrows \mathbb{R}^{d-1}$  defined as the normal cone to the ball  $B_{\sigma_{\text{yield}}} \subset \mathbb{R}^{d-1}$  of the radius  $\sigma_{\text{yield}}$  centered at the origin. The meaning of  $\bar{f}_\tau$  is the discrete driving force for the interfacial plasticity evolution. Fixing a time instant  $t$ , we can thus assume  $\bar{f}_\tau(t)$  bounded in  $L^\infty(\Gamma_C; \mathbb{R}^{d-1})$  and use (23) at time  $t$  tested by  $\bar{\pi}_\tau(t) - \pi(t)$  to execute the limit passage

$$\begin{aligned} \int_{\Gamma_C} \kappa_G |\nabla_s \bar{\pi}_\tau(t) - \nabla_s \pi(t)|^2 dS \\ = \int_{\Gamma_C} \left( \bar{f}_\tau(t) + \underline{\zeta}_\tau(t) \kappa_T [\bar{u}_\tau(t)]_T - \mathbb{T}\bar{\pi}_\tau(t) \right)^2 + \kappa_H |\bar{\pi}_\tau(t)|^2 \cdot (\bar{\pi}_\tau(t) - \pi(t)) \\ - \kappa_G \nabla_s \pi(t) : \nabla_s (\bar{\pi}_\tau(t) - \pi(t)) dS \rightarrow 0 \end{aligned} \quad (24)$$

where we again used the compact embedding  $H^1(\Gamma_C) \Subset L^2(\Gamma_C)$ . Thus the strong convergence (19c) follows.

The BV-functions (here in particular both BV-functions  $\zeta(\cdot)$  and  $\underline{\zeta}(\cdot)$ ) are continuous everywhere except at most countable number of times, let us denote this set of jumps

by  $J$ . Then we have  $\zeta(t) = \underline{\zeta}(t)$  for any  $t \in I \setminus J$ . In particular,  $\partial_u \mathcal{E}(t, u(t), \zeta(t), \pi(t)) = \partial_u \mathcal{E}(t, u(t), \underline{\zeta}(t), \pi(t)) \ni 0$  for such  $t$ , which proves (18a).

Now we can already pass to the limit in (18). The limit passage in (18a) for all  $t \in I \setminus J$  is simple just by continuity; note that we need  $\zeta(t) = \underline{\zeta}(t)$  for all  $t$  except from  $J$ . Thus (9a) is obtained.

For the limit passage in the semi-stability (18b) towards (9b), we use the so-called mutual recovery sequence

$$\tilde{\zeta}_\tau(x) := \begin{cases} \bar{\zeta}_\tau(t, x) \tilde{\zeta}(x) / \zeta(t, x) & \text{if } \zeta(t, x) > 0, \\ 0 & \text{if } \zeta(t, x) = 0. \end{cases} \quad (25)$$

with  $0 \leq \tilde{\zeta} \leq \zeta(t)$  given. After substituting  $\tilde{\zeta}_\tau$  in place of  $\tilde{\zeta}$  into (18b), we can easily pass to (9b) by continuity, namely

$$\begin{aligned} 0 &\leq \lim_{\tau \rightarrow 0} \int_{\Gamma_C} \underbrace{(\tilde{\zeta}_\tau - \bar{\zeta}_\tau(t))}_{\substack{\text{converges} \\ \text{weakly}^* \text{ in } L^\infty(\Gamma_C)}} \left( \underbrace{\frac{K_N}{2} [\bar{u}_\tau(t)]_N^2 + \frac{K_T}{2} |[\bar{u}_\tau(t)]_T - \mathbb{T} \bar{\pi}_\tau(t)|^2}_{\text{converges in } H^{1/2}(\Gamma_C) \subset L^1(\Gamma_C)} - a_1 \right) dS \\ &= \int_{\Gamma_C} (\tilde{\zeta} - \zeta(t)) \left( \frac{K_N}{2} [u(t)]_N^2 + \frac{K_T}{2} |[u(t)]_T - \mathbb{T} \pi(t)|^2 - a_1 \right) dS \end{aligned} \quad (26)$$

which is just the semistability (9b).

It is important that  $0 \leq \tilde{\zeta}_\tau \leq \bar{\zeta}_\tau(t)$  a.e. on  $\Gamma_C$  and, since  $\bar{\zeta}_\tau(t) \xrightarrow{*} \zeta(t)$ , also  $\tilde{\zeta}_\tau \xrightarrow{*} \tilde{\zeta}$  in  $L^\infty(\Gamma_C)$ . For this explicit construction (25), cf. also [19, Lemma 6.1] or [43, Formula (3.71)].

Also the limit passage in (18c) towards (9c) is simple just by continuity because we already proved the strong convergence (19c) otherwise the weak convergence would serve here too by semi-continuity arguments. The mutual recovery sequence can be even taken simply constant, namely  $\tilde{\pi}_\tau = \tilde{\pi}$ , so that:

$$\begin{aligned} &\int_{\Gamma_C} \zeta(t) \frac{K_T}{2} |[u(t)]_T - \mathbb{T} \pi(t)|^2 + \frac{K_H}{2} |\pi(t)|^2 + \frac{K_G}{2} |\nabla_s \pi(t)|^2 dS \\ &= \lim_{\tau \rightarrow 0} \int_{\Gamma_C} \bar{\zeta}_\tau(t) \frac{K_T}{2} |[\bar{u}_\tau(t)]_T - \mathbb{T} \bar{\pi}_\tau(t)|^2 + \frac{K_H}{2} |\bar{\pi}_\tau(t)|^2 + \frac{K_G}{2} |\nabla_s \bar{\pi}_\tau(t)|^2 dS \\ &\leq \lim_{\tau \rightarrow 0} \int_{\Gamma_C} \bar{\zeta}_\tau(t) \frac{K_T}{2} |[\bar{u}_\tau(t)]_T - \mathbb{T} \tilde{\pi}|^2 + \frac{K_H}{2} |\tilde{\pi}|^2 + \frac{K_G}{2} |\nabla_s \tilde{\pi}|^2 + \sigma_{\text{yield}} |\tilde{\pi} - \bar{\pi}_\tau(t)| dS \\ &= \int_{\Gamma_C} \zeta(t) \frac{K_T}{2} |[u(t)]_T - \mathbb{T} \tilde{\pi}|^2 + \frac{K_H}{2} |\tilde{\pi}|^2 + \frac{K_G}{2} |\nabla_s \tilde{\pi}|^2 + \sigma_{\text{yield}} |\tilde{\pi} - \pi(t)| dS. \end{aligned} \quad (27)$$

The limit passage in the energy (im)balance (18d) towards (9d) relies on the (strong $\times$ weak $\times$ strong)-continuity of  $\mathcal{E}(t, \cdot, \cdot, \cdot)$  on its definition domain. First we need to extend (18d) for all  $t_1$  and  $t_2$ . By (4a), we have  $\partial_t \mathcal{E}(t, \underline{u}_\tau, \underline{\zeta}_\tau, \underline{\pi}_\tau) = \langle \dot{f}_1, \underline{u}_\tau \rangle$ , and by the assumption  $f_1 \in W^{1,1}(I; H^1(\Omega \setminus \Gamma_C))$ , it is easy to see that

$$\begin{aligned} &\mathcal{E}(t_2, \bar{u}_\tau(t_2), \bar{\zeta}_\tau(t_2), \bar{\pi}_\tau(t_2)) + \mathcal{R}_0(\bar{\zeta}_\tau(t_2) - \bar{\zeta}_\tau(t_1)) + \text{Diss}_{\mathcal{R}_1}(\bar{\pi}_\tau; [t_1, t_2]) \\ &\leq \mathcal{E}(t_1, \bar{u}_\tau(t_1), \bar{\zeta}_\tau(t_1), \bar{\pi}_\tau(t_1)) - \int_{t_1}^{t_2} \langle \dot{f}_1, \underline{u}_\tau \rangle dt + \mathcal{O}(\tau), \end{aligned} \quad (28)$$

for all  $0 \leq t_1 \leq t_2 \leq T$ . By (19) and by the arguments we already used for (22), we can easily see that  $\mathcal{E}(t, \bar{u}_\tau(t), \bar{\zeta}_\tau(t), \bar{\pi}_\tau(t)) \rightarrow \mathcal{E}(t, u(t), \zeta(t), \pi(t))$ , which is to be used for (28) both for  $t = t_1$  and  $t = t_2$ .  $\square$

**Remark 3** (Approximate maximum-dissipation principle). One can devise the discrete analog of the integrated maximum-dissipation principle (13) straightforwardly for the left-continuous interpolants (15), required however to hold only asymptotically. More specifically, in analog to (13) formulated equivalently for all  $[0, t]$  instead of  $[t_1, t_2]$ , one can expect an *Approximate Maximum-Dissipation Principle* (AMDP) in the form

$$\int_0^t \bar{f}_\tau d\bar{\pi}_\tau \stackrel{?}{\sim} \text{Diss}_{\mathcal{R}_1}(\bar{\pi}_\tau; [0, t]) \quad \text{for some } \bar{f}_\tau \in -\partial_\pi \bar{\mathcal{E}}_\tau(\cdot, \bar{u}_\tau, \bar{\zeta}_\tau, \bar{\pi}_\tau), \quad (29a)$$

$$\int_0^t \bar{g}_\tau d\bar{\zeta}_\tau \stackrel{?}{\sim} \text{Diss}_{\mathcal{R}_0}(\bar{\zeta}_\tau; [0, t]) \quad \text{for some } \bar{g}_\tau \in -\partial_\zeta \bar{\mathcal{E}}_\tau(\cdot, \bar{u}_\tau, \bar{\zeta}_\tau, \bar{\pi}_\tau), \quad (29b)$$

where again the integrals are the lower Riemann-Stieltjes integrals as in (13) and where  $\bar{\mathcal{E}}_\tau(\cdot, u, \zeta, \pi)$  is the left-continuous piecewise-constant interpolant of the values  $\mathcal{E}(k\tau, u, \zeta, \pi)$ ,  $k = 0, 1, \dots, T/\tau$ . Moreover, “ $\stackrel{?}{\sim}$ ” in (29) means that the equality holds possibly only asymptotically for  $\tau \rightarrow 0$  but even this is rather only desirable and not always valid. Anyhow, loadings which, under given geometry of the specimen, lead to rate-independent slides where the solution is absolutely continuous will always comply with AMDP (29). Also, some finite-dimensional examples of “damageable springs” in [27, 39] show that this AMDP can detect too early rupturing local solutions (in particular the energetic ones) while it generically holds for solutions obtained by the algorithm (14). In our model, too early rupturing may also mean unphysical sliding into less dissipative Mode I even in situations when clearly Mode II should be active, cf. also the computational experiments in [51]. Generally speaking, (29) should rather be a-posteriori checked to justify the (otherwise not physically based) simple and numerically efficient fractional-step-type semi-implicit algorithm (14) from the perspective of the stress-driven solutions in particular situations and possibly to provide a valuable information that can be exploited to adapt time or space discretisation towards better accuracy in (29) and thus close towards the stress-driven scenario. Actually, for the piecewise-constant interpolants, we can simply evaluate the integrals explicitly, so that AMDP (29) reads

$$\sum_{k=1}^K \int_{\Gamma_C} \bar{f}_\tau^{k-1} (\pi_\tau^k - \pi_\tau^{k-1}) dS \stackrel{?}{\sim} \sum_{k=1}^K \int_{\Gamma_C} \sigma_{\text{yield}} |\pi_\tau^k - \pi_\tau^{k-1}| dS \quad \text{and} \quad (30a)$$

$$\sum_{k=1}^K \int_{\Gamma_C} \bar{g}_\tau^{k-1} (\zeta_\tau^k - \zeta_\tau^{k-1}) dS \stackrel{?}{\sim} \int_{\Gamma_C} a_1 (\zeta_0 - \zeta_\tau^K) dS \quad (30b)$$

where  $K = \max\{k \in \mathbb{N}; k\tau \leq t\}$  and

where  $\bar{f}_\tau^k \in -\partial_\pi \mathcal{E}(u_\tau^k, \zeta_\tau^{k-1}, \pi_\tau^k)$  and  $\bar{g}_\tau^k \in -\partial_\zeta \mathcal{E}(u_\tau^k, \zeta_\tau^k, \pi_\tau^k)$ .

Always, the left-hand sides in (30) are below the right-hand sides, and one can a-posteriori check the residua depending on  $t$  (or possibly also on space, cf. [51]).

## 5. NUMERICAL APPROXIMATION AND COMPUTATIONAL EXPERIMENTS

Let us assume  $\Omega \subset \mathbb{R}^d$  to be a polyhedral domain with  $\Gamma_D, \Gamma_N, \Gamma_C \subset \mathbb{R}^{d-1}$  also polyhedral. We outline briefly the discretisation by the finite-element method. In the simplest variant,  $\Omega$  is discretised by a triangular mesh  $\mathcal{T}_h$  consistently with the boundaries  $\Gamma_D$  and  $\Gamma_C$  with  $h > 0$  denoting the mesh parameter, and the polynomial P1-elements for  $u$ , P0-elements for  $\zeta$ , and P1-elements for  $\pi$  are employed. Applying such an approximation to (14), we thus arrive at two linear-quadratic programming problems:

$$\left. \begin{array}{ll} \text{minimize} & \mathcal{E}(k\tau, u, \zeta_\tau^{k-1}, \pi) + \mathcal{R}_1(\pi - \pi_\tau^{k-1}) \\ \text{subject to} & (u, \pi) \in H^1(\Omega \setminus \Gamma_C; \mathbb{R}^d) \times H^1(\Gamma_C; \mathbb{R}^{d-1}), \\ & (u, \pi) \text{ element-wise linear on } \mathcal{T}_h, \end{array} \right\} \quad (31a)$$

and, denoting the unique solution as  $(u_{\tau h}^k, \pi_{\tau h}^k)$ ,

$$\left. \begin{array}{ll} \text{minimize} & \mathcal{E}(k\tau, u_{\tau h}^k, \zeta, \pi_{\tau h}^k) + \mathcal{R}_0(\zeta - \zeta_{\tau h}^{k-1}) \\ \text{subject to} & \zeta \in L^\infty(\Gamma_C), \quad 0 \leq \zeta \leq 1, \\ & \zeta \text{ element-wise constant on } \mathcal{T}_h, \end{array} \right\} \quad (31b)$$

and denote its (possibly not unique) solution by  $\zeta_{\tau h}^k$ . Existence of such finite-dimensional solutions  $(u_{\tau h}^k, \zeta_{\tau h}^k, \pi_{\tau h}^k)$  is even simpler than in Section 4 because the considered linear spaces are finite-dimensional. Numerically, the solution can be obtained non-iteratively after a finite-number of steps if the linear-quadratic solver used for (31) is implemented in this way. More in detail,  $\mathcal{R}_1$  in the cost functional in (31a) is nonsmooth and, only after applying the Mosco-type transformation as e.g. in [37, Lemma 4], one obtains truly a quadratic programming problem (QP) if  $d = 2$  or a so-called second-order cone programming problem (SOCP) if  $d = 3$  when  $\mathcal{R}_1$  does not have a polyhedral graph, cf. e.g. [1, 46] for the SOCP algorithms.

**Proposition 3** (Unconditional convergence towards local solutions). Let again (10a-d) holds and let the spatial discretisation refines everywhere, i.e.  $\lim_{h \rightarrow 0} \sup_{\Delta \in \mathcal{T}_h} \text{diam}(\Delta) = 0$ . Then, the solution to the recursive alternating-minimization problem (31) exists and is numerically stable, i.e., in terms of the time-interpolants,

$$\|\bar{u}_{\tau h}\|_{L^\infty(I; H^1(\Omega \setminus \Gamma_C; \mathbb{R}^d))} \leq C, \quad (32a)$$

$$\|\bar{\zeta}_{\tau h}\|_{L^\infty(\Sigma_C) \cap \text{BV}(I; L^1(\Gamma_C))} \leq C, \quad (32b)$$

$$\|\bar{\pi}_{\tau h}\|_{L^\infty(I; H^1(\Gamma_C; \mathbb{R}^{d-1})) \cap \text{BV}(I; L^1(\Gamma_C; \mathbb{R}^{d-1}))} \leq C \quad (32c)$$

with some  $C$  independent of  $\tau > 0$  and  $h > 0$ . This solution satisfies the analog of (18) with the test functions  $\bar{u}$ ,  $\bar{\zeta}$ , and  $\bar{\pi}$  ranging over the above specified FEM-subspaces. Moreover, if  $\tau \rightarrow 0$  and  $h \rightarrow 0$ , then in terms of subsequences, like in Proposition 2, it converges to local solutions to the delamination problem (2)–(8):

$$\bar{u}_{\tau h}(t) \rightarrow u(t) \quad \text{in } H^1(\Omega \setminus \Gamma_C; \mathbb{R}^d) \quad \text{for all } t \in I, \quad (33a)$$

$$\bar{\zeta}_{\tau h}(t) \xrightarrow{*} \zeta(t) \quad \text{in } L^\infty(\Gamma_C) \quad \text{for all } t \in I, \quad (33b)$$

$$\bar{\pi}_{\tau h}(t) \rightarrow \pi(t) \quad \text{in } H^1(\Gamma_C; \mathbb{R}^{d-1}) \quad \text{for all } t \in I. \quad (33c)$$

*Sketch of the proof.* The arguments of the proof of Proposition 2 can be applied with only slight and mostly straightforward variation. Let us only briefly sketch differences beside that, of course, everywhere “ $\tau h$ ” is written in place of the subscript “ $\tau$ ” except in  $t_\tau$ .

The selection of converging subsequences is like in (20)–(21). Then, in (22), one must use an element-wise affine approximant of  $u$  rather than directly  $u$  itself. More in detail, as  $\bar{u}_{\tau h}(t) - u(t)$  is not a legal test function for the Galerkin analog of (18a), the estimate (22)



written with “ $\tau h$ ” in place of “ $\tau$ ” now modifies as

$$\begin{aligned}
& \int_{\Omega \setminus \Gamma_C} \mathbb{C} e(\bar{u}_{\tau h}(t) - u(t)) : e(\bar{u}_{\tau h}(t) - u(t)) \, dx \\
&= \int_{\Omega \setminus \Gamma_C} \mathbb{C} e(\bar{u}_{\tau h}(t) - u(t)) : e(\bar{u}_{\tau h}(t) - \bar{u}_{t,h}) + \mathbb{C} e(\bar{u}_{\tau h}(t) - u(t)) : e(\bar{u}_{t,h} - u(t)) \, dx \\
&\leq \int_{\Omega \setminus \Gamma_C} \mathbb{C} e(\bar{u}_{\tau h}(t) - u(t)) : e(\bar{u}_{\tau h}(t) - \bar{u}_{t,h}) + \mathbb{C} e(\bar{u}_{\tau h}(t) - u(t)) : e(\bar{u}_{t,h} - u(t)) \, dx \\
&\quad + \int_{\Gamma_C} \zeta_{\tau h}(t) \left( \kappa_N [\bar{u}_{\tau h}(t) - \bar{u}_{t,h}]_N^2 + \kappa_T [\bar{u}_{\tau h}(t) - \bar{u}_{t,h}]_T^2 \right) dS \\
&\leq \int_{\Omega \setminus \Gamma_C} \mathbb{C} e(u(t)) : e(\bar{u}_{t,h} - \bar{u}_{\tau h}(t)) + \mathbb{C} e(\bar{u}_{\tau h}(t) - u(t)) : e(\bar{u}_{t,h} - u(t)) \, dx \\
&\quad - \langle f_1(t_\tau), \bar{u}_{\tau h}(t) - \bar{u}_{t,h} \rangle + \int_{\Gamma_C} \zeta_{\tau h}(t) \left( \kappa_N [\bar{u}_{t,h}]_N \cdot [\bar{u}_{t,h} - \bar{u}_{\tau h}(t)]_N \right. \\
&\quad \left. + \kappa_T ([\bar{u}_{t,h}]_T - \mathbb{T} \bar{\pi}_{\tau h}(t)) \cdot [\bar{u}_{t,h} - \bar{u}_{\tau h}(t)]_T \right) dS \rightarrow 0 \quad (34)
\end{aligned}$$

where  $\bar{u}_{t,h}$  is element-wise linear on  $\mathcal{T}_h$  and approximates  $u(t)$  in the sense  $\bar{u}_{t,h} \rightarrow u(t)$  in  $H^1(\Omega \setminus \Gamma_C; \mathbb{R}^d)$ ; such  $\bar{u}_{t,h}$  always exists provided only  $h \rightarrow 0$  because the spatial discretisation is supposed to refine everywhere, and the possible dependence on the rate of approximation of  $u(t)$  on  $t$  is unimportant for (34).

Similarly, in (24), one must use an element-wise affine approximant of  $\pi$  rather than directly  $\pi$  itself. More in detail, (24) written with “ $\tau h$ ” in place of “ $\tau$ ” modifies as

$$\begin{aligned}
& \int_{\Gamma_C} \kappa_G |\nabla_s \bar{\pi}_{\tau h}(t) - \nabla_s \pi(t)|^2 dS = \int_{\Gamma_C} \kappa_G \nabla_s (\bar{\pi}_{\tau h}(t) - \pi(t)) : \nabla_s (\bar{\pi}_{\tau h}(t) - \bar{\pi}_{t,h}) \\
&\quad + \kappa_G \nabla_s (\bar{\pi}_{\tau h}(t) - \pi(t)) : \nabla_s (\bar{\pi}_{t,h} - \pi(t)) dS \\
&= \int_{\Gamma_C} \left( \bar{f}_{\tau h}(t) + \zeta_{\tau h}(t) \kappa_T [\bar{u}_{\tau h}(t)]_T - \mathbb{T} \bar{\pi}_{\tau h}(t) \right)^2 + \kappa_H |\bar{\pi}_{\tau h}(t)|^2 \cdot (\bar{\pi}_{\tau h}(t) - \bar{\pi}_{t,h}) \\
&\quad - \kappa_G \nabla_s \pi(t) : \nabla_s (\bar{\pi}_{\tau h}(t) - \bar{\pi}_{t,h}) + \kappa_G \nabla_s (\bar{\pi}_{\tau h}(t) - \pi(t)) : \nabla_s (\bar{\pi}_{t,h} - \pi(t)) dS \rightarrow 0 \quad (35)
\end{aligned}$$

where  $\bar{f}_{\tau h}(t)$  is the discrete driving force analogous as in (23) and again bounded in  $L^\infty(\Gamma_C; \mathbb{R}^{d-1})$ , and where  $\bar{\pi}_{t,h}$  is element-wise affine on  $\mathcal{T}_h$  and approximates  $\pi(t)$  in the sense  $\bar{\pi}_{t,h} \rightarrow \pi(t)$  in  $H^1(\Gamma_C; \mathbb{R}^{d-1})$ ; such  $\bar{\pi}_{t,h}$  always exists provided only  $h \rightarrow 0$  as the spatial discretisation refines everywhere, and again the possible dependence on the rate of approximation of  $\pi(t)$  on  $t$  is unimportant for (35).

Instead of (25), one can use the mutual recovery sequence:

$$\tilde{\zeta}_{\tau h}(x) := \begin{cases} \bar{\zeta}_{\tau h}(t, x) [\Pi_h^{(0)}(\tilde{\zeta}/\zeta(t))](x) & \text{if } [\Pi_h \zeta(t)](x) > 0, \\ 0 & \text{if } [\Pi_h \zeta(t)](x) = 0 \end{cases} \quad (36)$$

with  $\Pi_h^{(0)}$  denoting the element-wise constant interpolation on  $\Gamma_C$ , cf. also [26, Formula (4.35)]. If  $z(x) = 0$ , then also  $\tilde{z}(x) = 0$  because always  $0 \leq \tilde{z} \leq z$  and the fraction in (36) can be defined arbitrarily and valued in  $[0, 1]$ . The product of element-wise constant functions  $z_h$  and  $\Pi_h^{(0)}(\tilde{z}/z)$  is again element-wise constant, hence  $z_h \in Z_h$ . As  $0 \leq \Pi_h^{(0)}(\tilde{z}/z) \leq 1$ , we have also  $0 \leq \tilde{z}_h \leq z_h$ , hence  $\tilde{z}_h \in Z_h$  and  $\mathcal{B}_0(\tilde{z}_h - z_h) < \infty$ . As  $\Pi_h^{(0)}(\tilde{z}/z) \rightarrow \tilde{z}/z$  strongly in any  $L^p(\Gamma_C)$ ,  $p < +\infty$ , and  $z_h \xrightarrow{*} z$ ; here again we rely on that the spatial discretisation is supposed to refine everywhere. From (36) we have  $\tilde{z}_h \xrightarrow{*} z(\tilde{z}/z) = \tilde{z}$  in fact in  $L^\infty(\Gamma_C)$  due to the a priori bound of values in  $[0, 1]$ . The limit passage from the discretised analog of (18b) towards the semistability (9b) is then completely analogous to (26).

Also for the limit passage in the spatially-discretised analog of (18c), instead of just  $\tilde{\pi}$  fixed, one must use

$$\bar{\pi}_{\tau h} := \pi_{\tau h} - \Pi_h^{(1)}(\bar{\pi} - \pi) \quad (37)$$

with  $\Pi_h^{(1)}$  denoting the element-wise affine interpolation on  $\Gamma_c$ , cf. also [26, Formula (3.31)]. A modification of (27) is then straightforward because  $\tilde{\pi}_{\tau h} \rightarrow \tilde{\pi}$  strongly in  $H^1(\Gamma_c; \mathbb{R}^{d-1})$ ; also here we rely on that the spatial discretisation is supposed to refine everywhere.  $\square$

The approximate maximum-dissipation principle (30) now reads as:

$$\sum_{k=1}^K \int_{\Gamma_c} \mathfrak{f}_{\tau h}^{k-1} (\pi_{\tau h}^k - \pi_{\tau h}^{k-1}) \, dS \stackrel{?}{\sim} \sum_{k=1}^K \int_{\Gamma_c} \sigma_{\text{yield}} |\pi_{\tau h}^k - \pi_{\tau h}^{k-1}| \, dS \quad \text{and} \quad (38a)$$

$$\sum_{k=1}^K \int_{\Gamma_c} \mathfrak{g}_{\tau h}^{k-1} (\zeta_{\tau h}^k - \zeta_{\tau h}^{k-1}) \, dS \stackrel{?}{\sim} \int_{\Gamma_c} a_1 (\zeta_{\tau h}^K - \zeta_0) \, dS \quad (38b)$$

where  $\mathfrak{f}_{\tau h}^k \in -\partial_{\pi} \mathcal{E}(u_{\tau h}^k, \zeta_{\tau h}^{k-1}, \pi_{\tau h}^k)$  and  $\mathfrak{g}_{\tau h}^k \in -\partial_{\zeta} \mathcal{E}(u_{\tau h}^k, \zeta_{\tau h}^k, \pi_{\tau h}^k)$ , and  $K$  is as in (30).

It is a noteworthy attribute of our problem that all inelastic processes occur on the boundary  $\Gamma_c$  while in the bulk domains  $\Omega_1$  and  $\Omega_2$  it is linear. This allows for elimination of nodal values inside  $\Omega_1$  and  $\Omega_2$  and considerable reduction of degrees of freedom by considering only nodal or element values on  $\Gamma_c$ .

In fact, this idea has been systematically exploited even on the continuous level when implementing the boundary-element method, cf. [32, 41, 42, 48, 51], although it is still not fully supported by a convergence analysis like Corollary 3 due to general substantial theoretical difficulties related to this method.

Anyhow, for the computational experiments presented here with the goal to document rather modelling issues, we use a shortcut in implementing the spatial discretisation (31) by exploiting the collocation boundary-element method. Another numerical shortcut was neglecting the gradient term by putting  $\kappa_G = 0$ .

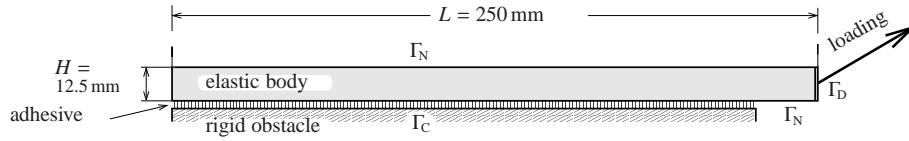


Fig. 1. Geometry and boundary conditions of the problem considered. The length of the initially glued part  $\Gamma_c$  is  $0.9L = 225$  mm, the adhesive layer has zero thickness.

We demonstrate varying mode-mixity of delamination on a relatively simple example motivated by the pull-push shear experimental test used in engineering practice [7]. Intentionally, we use the same geometry, shown in Fig. 1, as in [41] in order to compare our maximally-dissipative local solution with the energetic solution presented in [41]. In contrast to Sections 2–4, only one bulk domain is considered and  $\Gamma_c$  is a part of its boundary but this modification is straightforward; alternatively, one may also think about  $\Omega_2$  as a completely rigid body in the previous setting. Here  $\Omega_1$  is a two-dimensional rectangular domain glued on the most of its bottom side  $\Gamma_c$  with the Dirichlet loading acting on the right-hand side  $\Gamma_D$  in the direction  $(1, 0.6)$ , cf. Fig. 1, increasing linearly in time with velocity 1 mm/s.

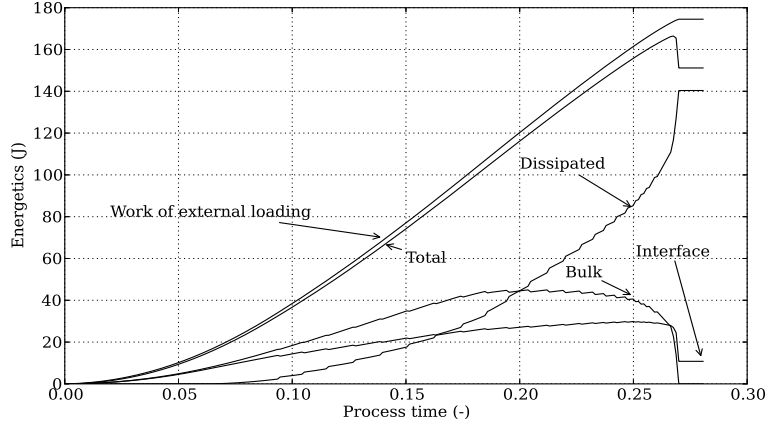


Fig. 2. Time evolution of the energies: the bulk and the interfacial parts of the stored energy  $\mathcal{E}(t, \bar{u}_{\tau h}(t), \bar{\zeta}_{\tau h}(t), \bar{\pi}_{\tau h}(t))$ , the dissipated energy  $\mathcal{R}_0(\bar{\zeta}_{\tau h}(t) - \zeta_0) + \text{Diss}_{\mathcal{Q}_1}(\bar{\pi}_{\tau h}; [0, t])$ , their sum = total energy (i.e. the left-hand side of (18d)), and the complementary work of external loading  $\int_0^t \langle \dot{f}_1, \underline{u}_{\tau h} \rangle dt$  (i.e. the right-hand side of (18d)).

The bulk material is considered isotropic homogeneous with the Young modulus  $E = 70$  GPa and Poisson's ratio  $\nu = 0.35$  (which corresponds to aluminum); thus  $\mathbb{C}_{ijkl} = \frac{\nu E}{(1+\nu)(1-2\nu)} \delta_{ij} \delta_{kl} + \frac{E}{2(1+\nu)} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$  with  $\delta_{ij}$  standing for the Kronecker symbol. For the adhesive, we took a normal stiffness  $\kappa_N = 150$  GPa/m, a tangential stiffness with  $\kappa_T = \kappa_N/2$ , the hardening slope  $\kappa_H = \kappa_T/9$ , and the Mode-I fracture toughness  $a_I = 187.5$  J/m<sup>2</sup>. The condition (5) here means  $2.65 \text{ MPa} < \sigma_{\text{yield}} < 5.3 \text{ MPa}$  and is indeed satisfied since  $\sigma_{\text{yield}} = 0.56 \sqrt{2\kappa_N a_I} = 0.56 \sqrt{4\kappa_T a_I} \cong 4.2 \text{ MPa}$ . This yields  $a_{II} \cong a_I + 629.1 \text{ J/m}^2 \cong 816.6 \text{ J/m}^2$ , the fracture-mode sensitivity  $a_{II}/a_I \cong 4.36$ ; cf. [41] for details. The initial conditions are, of course,  $\zeta_0 = 1$  and  $\pi_0 = 0$ ; the store energy  $\mathcal{E}(0, u_0, \zeta_0, \pi_0)$  is then 0.

It is interesting to check the energy (im)balance (18d). In Figure 2, we can see it depicted for  $t_1 = 0$  as a function of time  $t_2$ : the upper line is the right-hand side of (18d) while the line below is the left-hand side of (18d). We can clearly see that the difference is not zero and is increasing in time, which is in accord with (18d) because otherwise, if the difference would decrease on some time interval  $[t_1, t_2]$ , (18d) could not be valid on this interval. This non-vanishing difference between the left- and the right-hand sides of (18d) has, beside a possible numerical error, a physical meaning that some part of energy is lost (dissipated) due to rate-dependent mechanisms, which are neglected in the rate-independent model, like viscosity in the bulk, cf. [38, 42], or/and in the adhesive. One can thus expect that, if a (vanishing) viscosity would be considered e.g. in the bulk, the defect measure arising by this mechanism (like that one calculated in [42]) would likely have the overall energy corresponding just to this gap. Also note that, after the complete delamination, the stored energy in the adhesive (interface) does not vanish due to the energy deposited into the hardening.

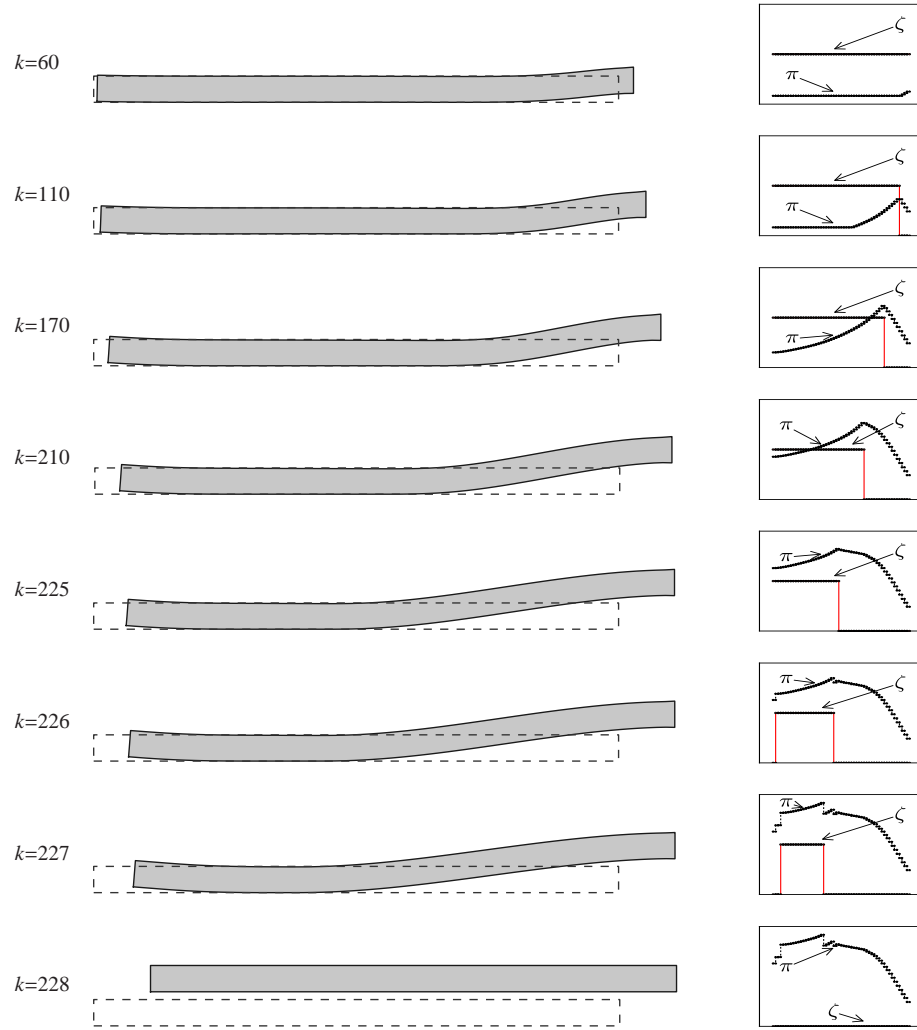


Fig. 3. Time evolution at eight snapshots of the geometrical configuration (displacement depicted magnified  $100\times$ ) and the spatial distribution of  $\zeta$  and  $\pi$  along  $\Gamma_C$ .

This example exhibits remarkably varying mode of delamination. At the beginning the delamination is performed by a mixed mode close to Mode I given essentially by the direction of the Dirichlet loading, cf. Figure 1, while later it turns rather to nearly pure Mode II. Yet, at the very end of the process, due to elastic bending the delamination starts performing also from the left-hand side of the bar opposite to the loading side, and thus again a mixed mode occurs. This relatively complicated mixed-mode behaviour is depicted in Figures 3–4, showing essential qualitative difference from the energetic solution which exhibits a non-physical tendency to slide to less-dissipative Mode I, cf. [41, Fig. 7].

The evolution of the deformation  $u$  and spatial distribution of the delamination  $\zeta$  and the plastic slip  $\pi$  are depicted in Figure 3 at eight snapshots selected not uniformly to visualize interesting effects when delamination starts to be completed. In particular, the delamination propagating from both sides at the very end (mentioned already above) is seen there.

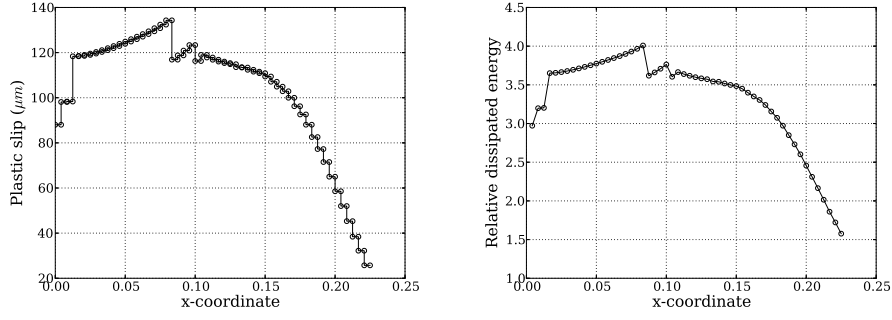


Fig. 4. Distribution of mode-mixity of delamination along  $\Gamma_C$ :  
 Left: The overall plastic slip after the delamination has been completed (=the last snapshot in Figure 3).  
 Right: The dissipated energy related to  $a_i$  after the delamination has been completed (value=1 ~ Mode I, value=3.97 ~ Mode II). Similar distributions are observed in both plots because there were not cycling in plastification during the delamination.

For the discretisation of the experiment in Figures 2–4, we choose  $\tau = 0.012$  and  $h = 4.6$  mm (=the size of a boundary element in uniform discretisation).

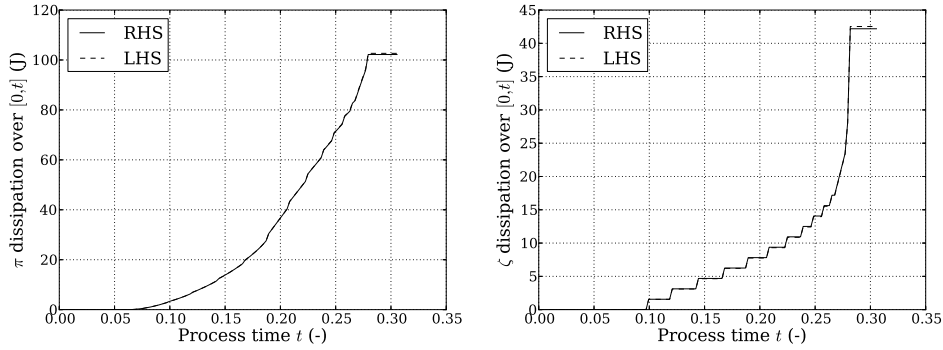


Fig. 5. Time evolution of the left- and the right-hand sides in the approximate maximum dissipation principle (AMDP) for the plastic slip  $\pi$ , i.e. (38a), and the damage parameter  $\zeta$ , i.e. (38b). The difference is practically invisible in the former case and less than 2% in the latter case.

The differences in the approximate maximum-dissipation principles (38) are now displayed in Figure 5. We can see that our algorithm yielded a well (about 98%) maximally-dissipative (i.e. stress-driven) solution, the possible deviation is possibly only in  $\zeta$  at the very end of the delamination process.

Eventually, the joint convergence from Corollary 3 for time- and FEM-spatial discretisation (although here implemented by BEM) is demonstrated in Figures 6 and 7 for a twice coarser time/space discretisations. We choose the scenario keeping the ratio  $\tau/h$  constant, although Corollary 3 itself does not give any particular suggestion in this respect. Anyhow, the tendency of convergences is clearly seen, although we naturally do not know the exact solution so that we cannot evaluate any actual error. On top of it, the exact solution does not need to be unique so we even do not have guaranteed the convergence of the whole sequence of the approximate solutions and, moreover, the simplified implementation by collocation BEM does not have guaranteed convergence, in contrast to FEM stated in Corollary 3.

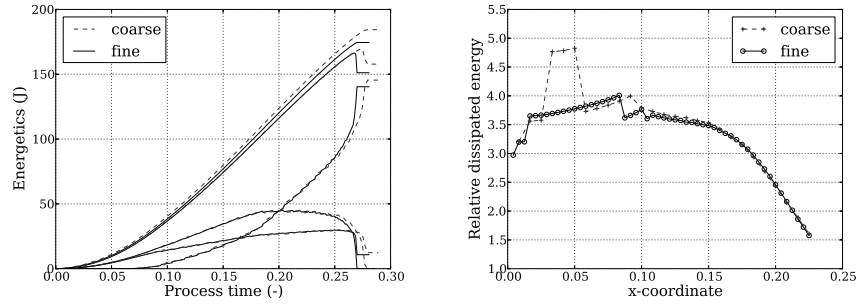


Fig. 6. Convergence test: Left: evolution of energies as in Figure 2  
Right: final spatial distribution of  $\pi$  along  $\Gamma_c$  as in Figure 4(right).

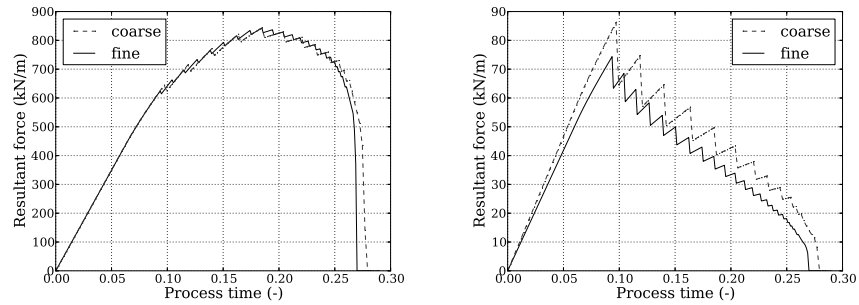


Fig. 7. Convergence test: horizontal (left) and vertical (right) component of the total force response evolving in time.

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